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Objective

The notion of the **central path** plays an important role in the development of primal-dual interior-point algorithms.

Our objective in this work is to show that a related notion called the **quasicentral path (QCP)**, introduced by Argáez and Tapia [2, 3] in nonlinear programming, while being a less restrictive notion it is sufficiently strong to guide the iterates towards a solution of the Linear Programming problem.
**Problem Formulation: Linear Programming Problem**

**Primal Problem (1)**

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{subject to} & \quad Ax = b, \quad x \geq 0
\end{align*}
\]

Where

- \( c, x \in \mathbb{R}^n \)
- \( b \in \mathbb{R}^m \)
- \( A \in \mathbb{R}^{m \times n} \)

**Dual Problem (2)**

\[
\begin{align*}
\text{max} & \quad b^T y \\
\text{subject to} & \quad A^T y + z = c, \quad z \geq 0
\end{align*}
\]

Where

- \( y \in \mathbb{R}^m \)
- \( z \in \mathbb{R}^n \)

\( A \) is full rank and \((m<n)\)
Optimally Conditions

KKT (Necessary and Sufficient) Conditions

\[ F : \mathbb{R}^{n+m+n} \rightarrow \mathbb{R}^{m+n+n} \]

\[ F(x, y, z) = \begin{cases} 
Ax-b \\
A^T y + z - c \\
XZe \\
(x, z) \geq 0
\end{cases} = 0 \quad (3) \]

Where \( X = \text{diag}(x), \ Z = \text{diag}(z), \) and \( e = (1, \ldots, 1)^T \in \mathbb{R}^n \)

Remark: Newton's method applies to KKT Conditions preclude before obtain a solution of the problem.

Perturbed KKT Conditions

For \( \mu > 0, \)

\[ F_\mu : \mathbb{R}^{n+m+n} \rightarrow \mathbb{R}^{m+n+n} \]

\[ F_\mu (x, y, z) = \begin{cases} 
Ax-b \\
A^T y + z - c \\
XZe - \mu e \\
(x, z) \geq 0
\end{cases} = 0 \quad (4) \]

Property: Under some assumptions for each \( \mu > 0, \) the perturbed system has a unique solution \( (x(\mu), y(\mu), z(\mu)) \). Moreover, \( (x(\mu), y(\mu), z(\mu)) \) tends to a solution \( (x, y, z) \) to the primal and dual problem as \( \mu \rightarrow 0. \)
Fundamental Properties

Property 1:

\[ e_p^k = b - Ax_k = (1 - \alpha_k) e_p^{k-1} = \prod_{j=1}^{k} (1 - \alpha_j) e_p^0 \]
\[ e_d^k = c - A^T y_k - z_k = (1 - \alpha_k) e_d^{k-1} = \prod_{j=1}^{k} (1 - \alpha_j) e_d^0 \]

\[ 0 < \alpha_j \leq 1 \]

Property 2 (see [1]):

If \[ \| e_d^0 \| \leq \| e_p^0 \| \] then \[ \| e_d^k \| \leq \| e_p^k \| \] \[ \forall_k \]

This means: \[ e_d = 0 \] if \[ e_p = 0 \]
Globalization Strategy

Central Path (Classical Approach)

- $Ax - b = 0$
- $A^T y + z - c = 0$
- $XZe - \mu e = 0$
- $(x, z) \geq 0$

QCP (Our Approach)

- $Ax - b = 0$
- $v/s$
- $XZe - \mu e = 0$
- $(x, z) \geq 0$
We propose to follow the QCP as a globalization strategy using as initial point \((x_0, y_0, z_0)\) such that \(e_d^0 \leq e_p^0\).

In this situation, then we can remove the dual condition from the central path, and consider the QCP as a central region to be followed for obtaining a solution of the primal and dual problems simultaneously.

To make progress to the QCP, we present a new merit function.
Progress to the QCP: Sufficient Decrease

Merit Function
To progress to the QCP we use the following function

For $\mu > 0$,

$$
\Phi_\mu : \mathbb{R}^{n+n} \rightarrow \mathbb{R}
$$

$$
\Phi_\mu (x, z) = \frac{1}{2} \|Ax - b\|^2 + \sum_{i=1}^{n} (x_iz_i - \mu \ln(x_iz_i))
$$

Property: Descent direction

$$
\nabla \Phi_\mu (x, z)^T \begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix} = -\left( \|Ax - b\|^2 + \|W(xZe - \mu e)\|^2 \right) < 0
$$

This means at each Newton direction $(\Delta x, \Delta z)$ we can progress toward the QCP
The Algorithm

**Step 1.** Consider an initial point \( x>0, z>0, y=0 \) such that \( \|e_d\| \leq \|e_p\| \)

Choose \( \mu>0, \gamma, \tau, \sigma \in (0,1) \)

Set \( e_d = c-z, \ e_p = b-Ax, \ e_c = \mu e - XZe \)

**Step 2.** Newton direction. Solve the linear system for \( (\Delta x, \Delta y, \Delta z) \)

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & A^T & I \\
Z & 0 & X
\end{pmatrix}\begin{pmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{pmatrix} = -\begin{pmatrix}
Ax - b \\
A^T y - c - z \\
XZe - \mu e
\end{pmatrix}
\]

**Step 3.** Forcing positivity for \( x \) and \( z \). Calculate \( \tilde{\alpha} = \min \{1, \tau \hat{\alpha} \} \)

Where \( \hat{\alpha} = \frac{-1}{\min(X^{-1}\Delta x, Z^{-1}\Delta z)} \)

Such that \( x + \tilde{\alpha} \Delta x > 0 \) and \( z + \tilde{\alpha} \Delta z > 0 \)
Step 4. Progress to the QCP (Line search). Find $\alpha = \left( \frac{1}{2} \right)^t \tilde{\alpha}$ where $t$ is the smallest positive integer such that

$$\Phi_\mu \left( x + \alpha \Delta x, z + \alpha \Delta z \right) \leq \Phi_\mu \left( x, z \right) + 10^{-4} \alpha \nabla \Phi_\mu \left( x, z \right)^T \left( \Delta x, \Delta z \right)$$

Step 5. Update

$$x = x + \alpha \Delta x > 0 \quad e_p = (1 - \alpha) e_p$$
$$z = z + \alpha \Delta z > 0 \quad e_d = (1 - \alpha) e_d$$

Step 6. Proximity to QCP. If

$$\left( \| e_p \|^2 + \left\| (XZ)^{-1/2} (XZe - \mu e) \right\|^2 \right) \leq \gamma \mu$$

Then go to Step 7. Else, set $e_c = \sigma \mu e - XZe$ and go to step 2.

Step 7. Stopping criteria. If

$$\left( 2 \| e_p \| + x^T z \right) \left( 1 + \| p \| \right) < \varepsilon$$

then stop. Else, update $\mu$, set $e_c = \sigma \mu e - XZe$ and go to Step 2.
Numerical Results

Problem No 1 "25FV47" taken of the Netlib Test Problems:
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Problems were taken of the Netlib Test Problems:

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The Notion of the Quasicentral Path in Linear Programming

References


Acknowledgments

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