Computational Science

The Notion of the Quasicentral Path in Linear Programming

Carlos Andres Ramirez

PhD. Computational Science Student

Advisors

Dr. Miguel Argaez

Dr. Leticia Velazquez

April 16, 2009

The University of Texas at El Paso

Objective

The notion of the **central path** plays an important role in the development of primal-dual interior-point algorithms.

Our objective in this work is to show that a related notion called the **quasicentral path (QCP)**, introduced by Argáez and Tapia [2, 3] in nonlinear programming, while being a less restrictive notion it is sufficiently strong to guide the iterates towards a solution of the Linear Programming problem.

Problem Formulation: Linear Programming Problem

Primal Problem (1)

$$\boxed{\min c^T x \quad \text{s.t}\; Ax = b, \quad x \geq 0}$$

Where

$$c, x \in \mathbb{R}^n,\; b \in \mathbb{R}^m,\; A \in \mathbb{R}^{m \times n}$$

A is full rank and *(m<n)*

Dual Problem (2)

$$\boxed{\max b^T y \quad \text{s.t}\; A^T y + z = c, \quad \text{z} \geq 0}$$

Where

$$y \in \mathbb{R}^m,\; z \in \mathbb{R}^n$$

Optimally Conditions

KKT (Necessary and Sufficient) Conditions

$$F:\mathbb{R}^{n+m+n}\to\mathbb{R}^{m+n+n}$$

$$F(x,y,z)=\begin{pmatrix} Ax-b \\ A^T y+z-c \\ XZe \end{pmatrix}=0 \qquad (3)$$

$$(x,z)\geq 0$$

Where $X=diag(x),\ Z=diag(z)$, and $e=(1,\ldots,1)^T\in\mathbb{R}^n$

Remark: Newton's method applies to KKT Conditions preclude before obtain a solution of the problem.

Perturbed KKT Conditions

For μ>0,

$$F_\mu:\mathbb{R}^{n+m+n}\to\mathbb{R}^{m+n+n}$$

$$F_\mu(x,y,z)=\begin{pmatrix} Ax-b \\ A^T y+z-c \\ XZe-\mu e \end{pmatrix}=0 \qquad (4)$$

$$(x,z)\geq 0$$

Property : Under some assumptions for each μ>0, the perturbed system has a unique solution $(x(\mu),y(\mu),z(\mu))$. Moreover, $(x(\mu),y(\mu),z(\mu))$ tends to a solution (x,y,z) to the primal and dual problem as $\mu\to 0$.

Fundamental Properties

Property 1:

$$e_p^k = b - Ax_k = (1-\alpha_k)\, e_p^{k-1} = \prod_{j=1}^{k}(1-\alpha_j)\, e_p^0$$

$$e_d^k = c - A^T y_k - z_k = (1-\alpha_k)\, e_d^{k-1} = \prod_{j=1}^{k}(1-\alpha_j)\, e_d^0$$

$$0 < \alpha_j \le 1$$

Property 2 (see [1]):

If $\|e_d^0\| \le \|e_p^0\|$ then $\|e_d^k\| \le \|e_p^k\| \;\; \forall_k$

This means: $e_d = 0 \quad \text{if} \quad e_p = 0$

Globalization Strategy

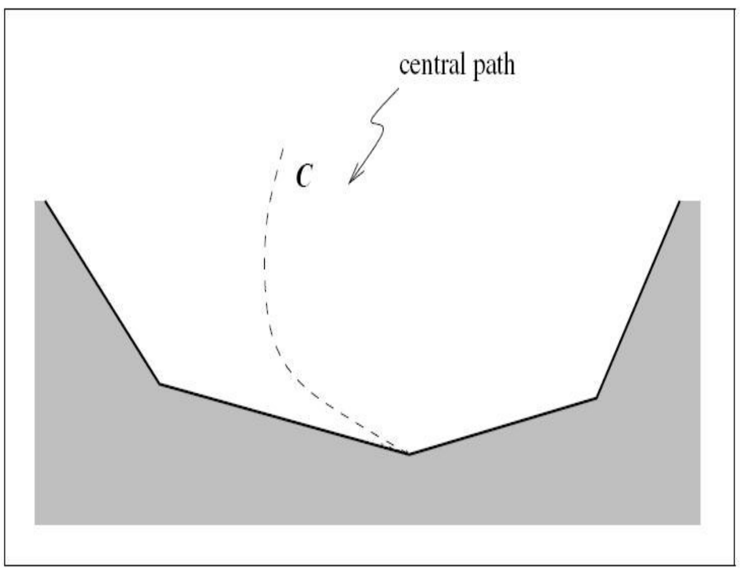
Central Path (Classical Approach)

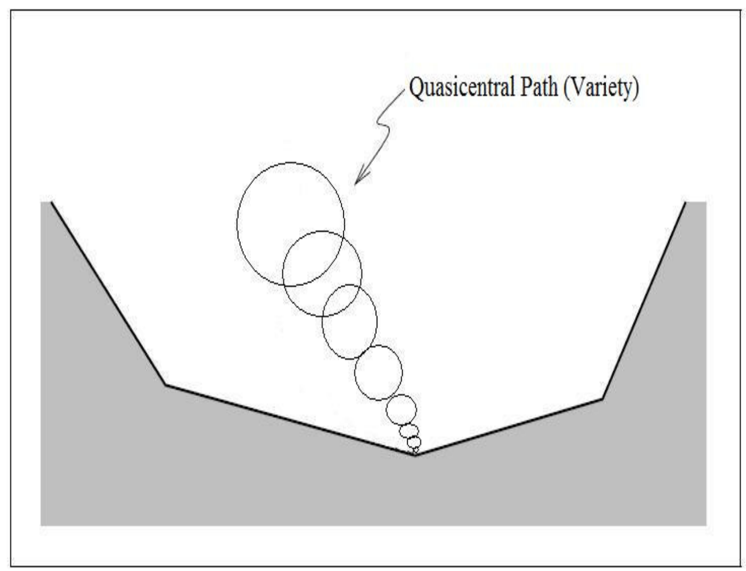
$$Ax - b = 0$$
$$A^T y + z - c = 0$$
$$XZe - \mu e = 0$$
$$(x, z) \geq 0$$

v/s

QCP (Our Approach)

$$Ax - b = 0$$
$$XZe - \mu e = 0$$
$$(x, z) \geq 0$$





We propose to follow the QCP as a globalization strategy using as initial point (x_0, y_0, z_0) such that $\|e_d^0\| \le \|e_p^0\|$

In this situation, then we can remove the dual condition from the central path, and consider the QCP as a central region to be followed for obtaining a solution of the primal and dual problems simultaneously

To make progress to the QCP, we present a new merit function

Progress to the QCP: Sufficient Decrease

Merit Function

To progress to the QCP we use the following function

For $\mu > 0$,

$$\Phi_\mu : \mathbb{R}^{n+n}_{++} \to \mathbb{R}$$

$$\Phi_\mu(x,z) = \frac{1}{2}\|Ax-b\|^2 + \sum_{i=1}^{n}\left(x_i z_i - \mu \ln\left(x_i z_i\right)\right)$$

Property: Descent direction

$$\nabla\Phi_\mu\left(x,z\right)^T\begin{pmatrix}\Delta x\\ \Delta z\end{pmatrix} = -\left(\|Ax-b\|^2 + \|W\left(XZe-\mu e\right)\|^2\right) < 0$$

This means at each Newton direction $\left(\Delta x, \Delta z\right)$ we can progress toward the QCP

The Algorithm

Step 1. Consider an initial point *x>0, z>0, y=0* such that $\|e_d\| \le \|e_p\|$

Choose $\mu > 0, \quad \gamma, \tau, \sigma \in (0,1)$

Set $e_d = c - z,\ e_p = b - Ax,\ e_c = \mu e - XZe$

Step 2. Newton direction. Solve the linear system for $(\Delta x, \Delta y, \Delta z)$

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y - c - z \\ XZe - \mu e \end{pmatrix}$$

Step 3. Forcing positivity for *x* and *z*. Calculate $\tilde{\alpha} = \min(1, \tau\hat{\alpha})$

Where $\hat{\alpha} = \dfrac{-1}{\min(X^{-1}\Delta x, Z^{-1}\Delta z)}$

Such that $x + \tilde{\alpha}\Delta x > 0 \quad \text{and} \quad z + \tilde{\alpha}\Delta z > 0$

Step 4. Progress to the QCP (Line search). Find $\alpha = \left(\frac{1}{2}\right)^t \tilde{\alpha}$ where *t* is the smallest positive integer such that

$$\Phi_\mu\left(x+\alpha\Delta x, z+\alpha\Delta z\right) \le \Phi_\mu\left(x,z\right) + 10^{-4}\alpha\nabla\Phi_\mu\left(x,z\right)^T\left(\Delta x, \Delta z\right)$$

Step 5. Update $x = x+\alpha\Delta x > 0$ $\quad e_p = (1-\alpha)e_p$

$z = z+\alpha\Delta z > 0$ $\quad e_d = (1-\alpha)e_d$

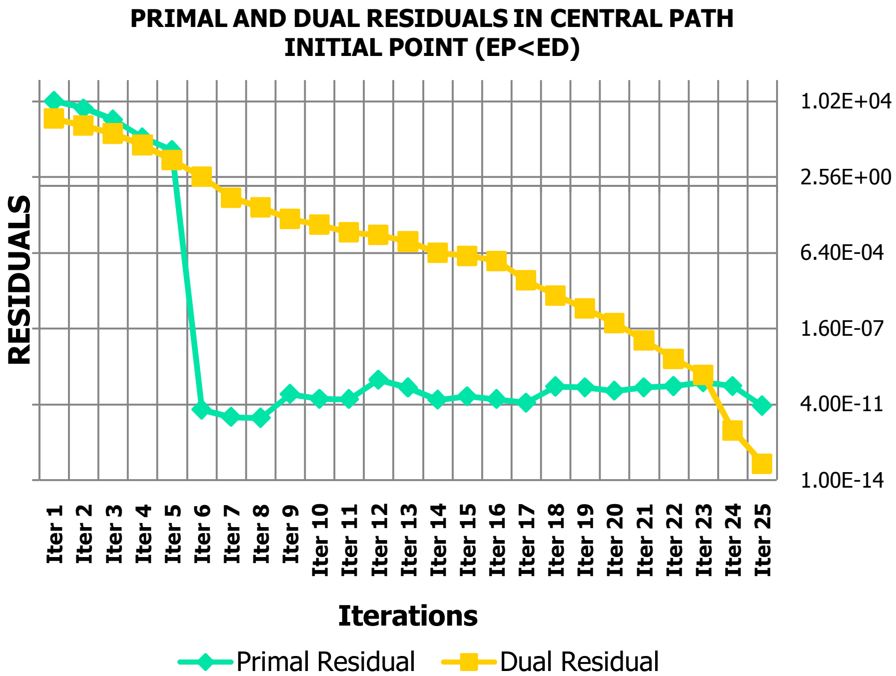
Step 6. Proximity to QCP. If $\left(\left\|e_p\right\|^2 + \left\|(XZ)^{-1/2}\left(XZe-\mu e\right)\right\|^2\right) \le \gamma\mu$

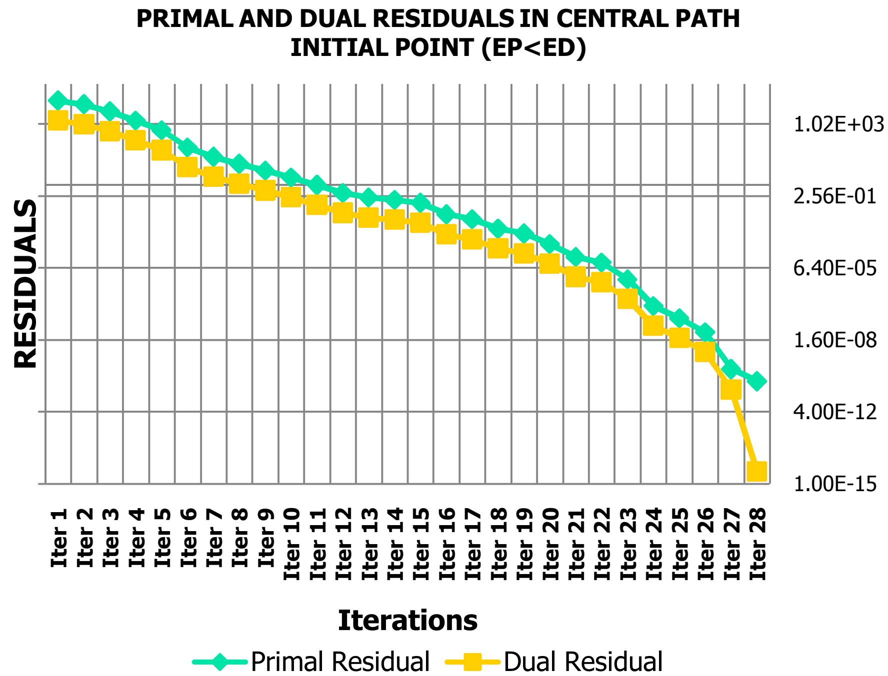
Then go to Step 7. Else, set $e_c = \sigma\mu e - XZe$ and go to step 2

Step 7. Stopping criteria. If $\left(2\left\|e_p\right\| + x^T z\right) \big/ \left(1+\left\|b\right\|\right) < \varepsilon$ then stop

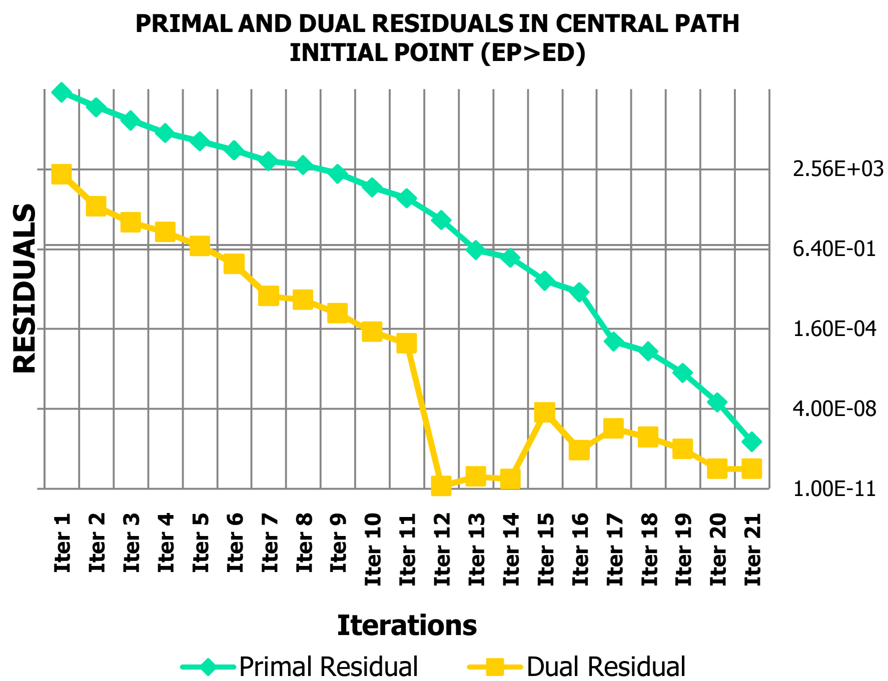
Else, update μ, set $e_c = \sigma\mu e - XZe$ and go to Step 2

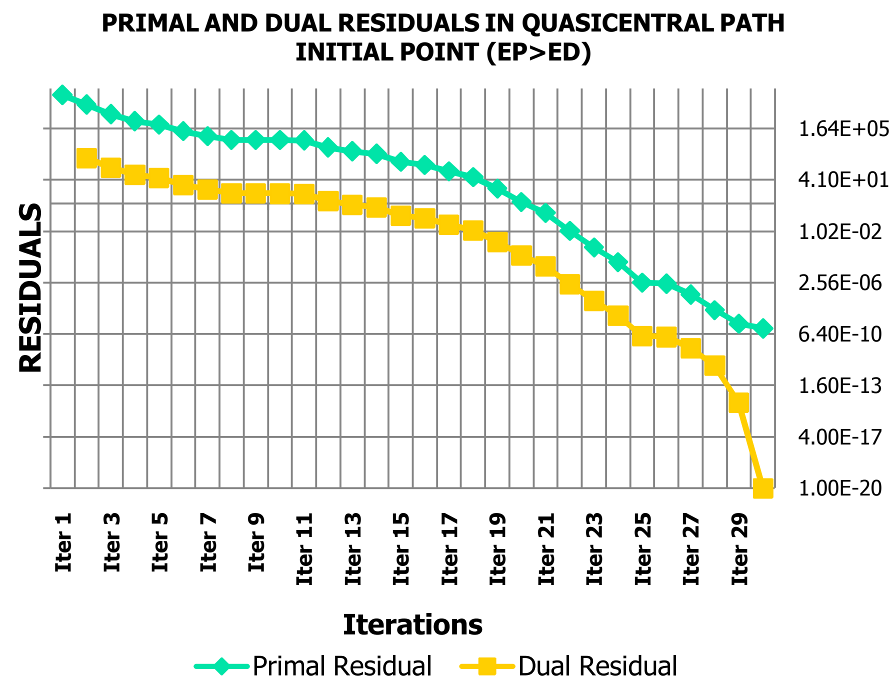
Numerical Results



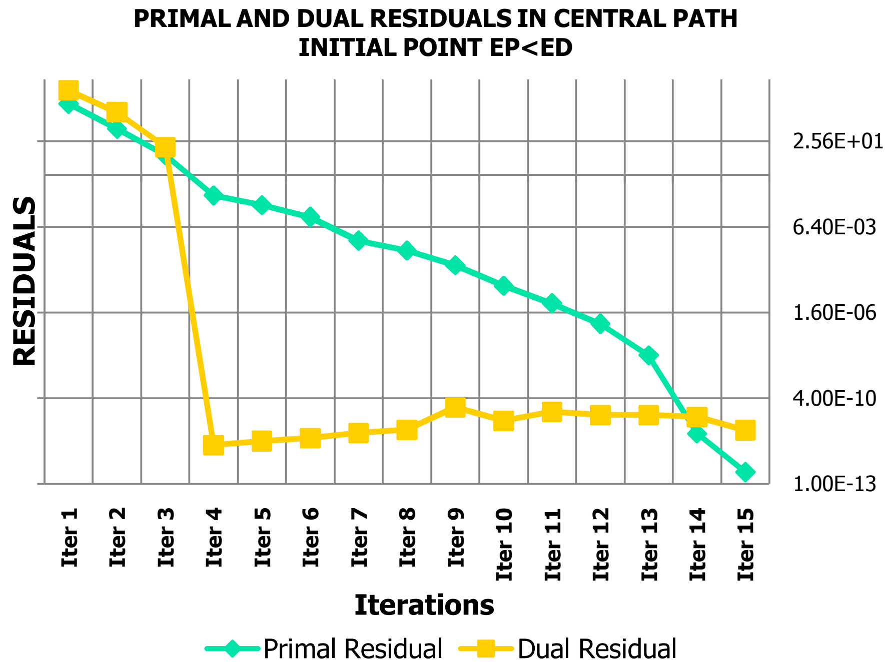


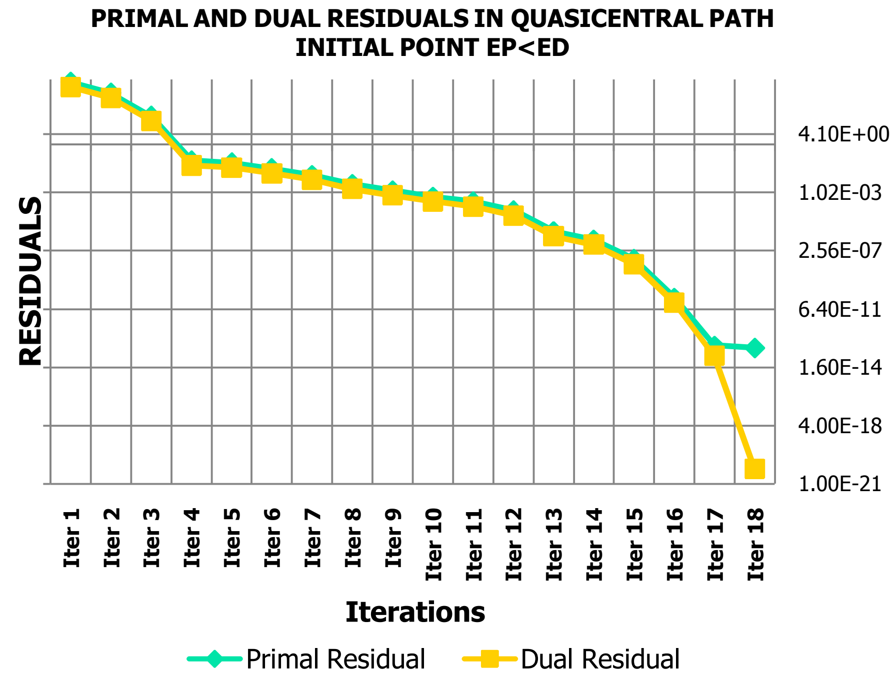
Problem No 1 "25FV47" taken of the Netlib Test Problems:
http://www-fp.mcs.anl.gov/OTC/Guide/TestProblems/LPtest/



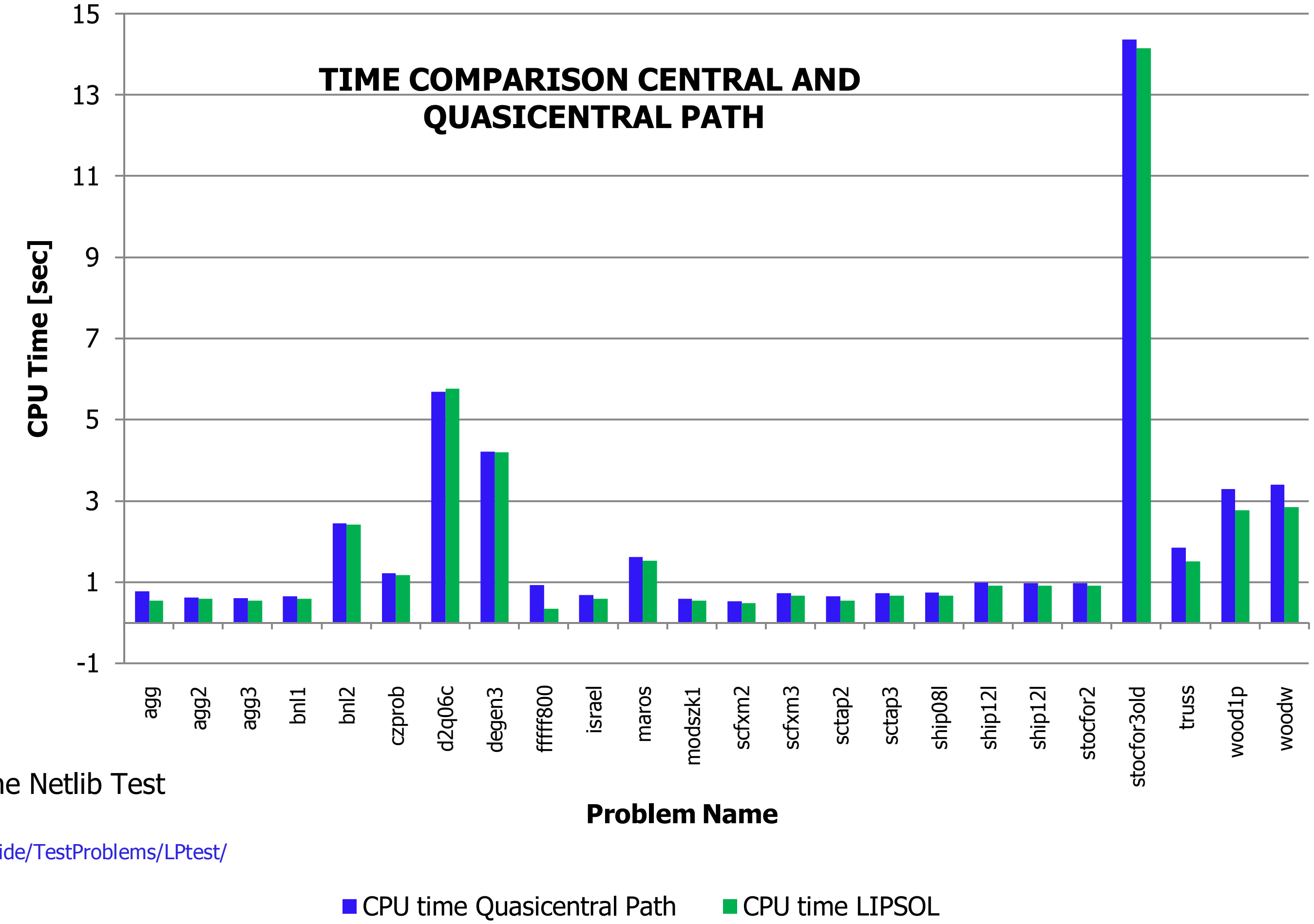


Problem No 5 “agg” taken of the Netlib Test Problems:
http://www-fp.mcs.anl.gov/OTC/Guide/TestProblems/LPtest/





Problem No 65 "scorpion" taken of the
Netlib Test Problems:
http://www-fp.mcs.anl.gov/OTC/Guide/TestProblems/LPtest/



Problems were taken of the Netlib Test Problems:
http://www-fp.mcs.anl.gov/OTC/Guide/TestProblems/LPtest/

Problem Features			Quasicentral Algorithm			LIPSOL Algorithm		
Problem Name	**m**	**n**	**Iter**	**CPU time**	**Primal residual**	**Iter**	**CPU time**	**Primal residual**
scagr7	129	185	19	0.15	8.17E-09	14	0.12	2.08E-11
scfxm1	322	592	21	0.32	6.77E-09	19	0.27	8.54E-15
scfxm2	644	1184	21	0.52	9.39E-09	21	0.47	1.96E-14
scfxm3	966	1776	21	0.72	5.93E-09	21	0.66	1.33E-13
scorpion	375	453	18	0.2	2.60E-13	15	0.16	3.25E-13
scrs8	485	1270	25	0.41	6.56E-09	24	0.37	4.73E-12
scsd1	77	760	11	0.17	5.82E-09	9	0.11	8.07E-09
scsd6	147	1350	11	0.21	8.26E-09	11	0.18	1.23E-09
scsd8	397	2750	10	0.28	5.06E-09	11	0.31	1.43E-11
sctap1	300	660	19	0.23	6.53E-09	17	0.19	4.5E-13
sctap2	1090	2500	20	0.64	6.27E-09	19	0.53	5.03E-13
sctap3	1480	3340	18	0.72	3.15E-09	18	0.66	1.6E-14
share1b	112	248	24	0.23	6.60E-09	22	0.2	5.28E-15
share2b	96	162	17	0.14	5.68E-09	13	0.1	9.01E-13
ship04l	356	2162	14	0.36	7.29E-09	14	0.32	3.69E-14
ship04s	268	1414	17	0.3	5.88E-09	14	0.26	1.09E-10
ship08l	688	4339	17	0.74	4.93E-09	16	0.66	1.1E-12
ship08s	416	2171	16	0.41	5.86E-09	15	0.34	2.98E-13
ship12l	838	5329	19	0.97	3.11E-09	18	0.9	5E-11
ship12l	838	5329	19	0.98	3.11E-09	18	0.9	5E-11
stocfor1	109	157	20	0.15	7.47E-09	16	0.14	2.39E-11
stocfor2	2157	3045	21	0.96	9.57E-09	21	0.9	1.85E-10
stocfor3old	16675	23541	35	14.37	6.57E-09	35	14.16	2.06E-09
truss	1000	8806	22	1.85	2.60E-09	19	1.51	3.69E-11
wood1p	244	2595	22	3.29	7.71E-09	19	2.77	3.23E-09
woodw	1098	8418	31	3.39	7.81E-09	28	2.84	6.67E-10

Problems were taken of the Netlib Test Problems:
http://www-fp.mcs.anl.gov/OTC/Guide/TestProblems/LPtest/

The Notion of the Quasicentral Path in Linear Programming

References

[1] M. Argáez, O. Mendez, L. Velazquez, The Notion of the Quasicentral Path in Linear Programming. Submitted to SIAM Journal on Optimization (2008).

[2] M. Argáez & R. A. Tapia, On the global convergence of a modified augmented Lagrangian linesearch interior-point Newton method for nonlinear programming, J*. Optima. Theory Appl.*, 114 (2002), 1-25.

[3] M. Argáez, Exact and inexact Newton linesearch interior-point algorithms for nonlinear programming problems, PhD Thesis, Department of Computational and Applied Mathematics, Rice University, Houston, Texas, (1997).

[4] Zhang Yin, Solving Large-Scale Linear Programs by Interior-Point Methods Under the MATLAB Environment (1996).

Acknowledgments

The authors thank the financial support from the Computational Science Program and ARL Grant No W911NF-07-2-0027.

The authors also acknowledge the office space provided by the Department of Mathematical Sciences.